Automated Study of Envelopes: the transition from 1-parameter to 2-parameter families of surfaces

Thierry Dana-Picard
e-mail: ndp@jct.ac.il
Jerusalem College of Technology
Jerusalem, Israel

Nurit Zehavi
e-mail: nurit.zehavi10@gmail.com
Weizman Institute for Science
Rehovot, Israel

Abstract

The study of parametrized families of curves and surfaces is a classical topic of great importance in applied science and engineering. It suffers from a lack of rigorous theory and of theorems. The usage of technology such as Computer Algebra Systems (CAS) may give this mathematical domain a new role in STEM education.

Two central features of technology are used here: the graphical register of the CAS, and the algebraic algorithms provide automated proof of the results.

Their respective roles are different when working in 2D and in 3D. In this last case, the joint influence of visualization problems and the non-availability of certain tools such as a slider push the central aspects of the study towards automated proofs.

1. Introduction

A long time ago, some classical topics disappeared from the curriculum in Differential Geometry. In 1962, Thom complained about the disappearance of envelopes of parameterized families of curves and surfaces (see [19]). Nevertheless, the topic was still included in textbooks, such as [4], but nor for a long time. Later among the possible reasons for that disappearance are the facts that the theory is not so well-developed, that numerous theorems do not exist but numerous special cases exist. Moreover, the geometric nature of the topic induces the need for visualization skills which are not frequent. This disappearance is problematic, as envelopes of curves and more than that of surfaces are ubiquitous in industrial plants. The topic is present in robotics, optics, ballistics, and numerous other applied fields, making it fully relevant to STEM (Science, Technology, Engineering and Mathematics) education.

Trigueros-Gaismann and Martinez-Planell note in [20] that research on particularities of multivariable functions to explicitly study how students build their understanding of them is scarce. This problem is even stronger when studying general surfaces in 3-dimensional space: according to the theorem of implicit functions, a surface given by an implicit equation may be viewed *locally* as the graph of a two-variable function, but this may not help a freshman to grasp the surface globally. This theorem is useful for studying local properties, but the existence of envelopes is a global property of a family of surfaces. The visualization issue is harder in this framework.

In the same paper, they note that, in order to understand functions, it is important to relate different registers of representation; see [9]. Switching between different registers of representation is an important skill in the study of envelopes of parametrized families of plane curves presented in [7]. In [21], Yerushalmy points out the importance of the interplay between different registers of representation for the transition from the study of one-variable to the study of two-variable functions. When beginning working on envelopes of parametrized families of surfaces, we could rely on the assumption that this interplay will be even more important, as surfaces may be represented in more ways than plane curves and as the visualization issue is harder in this case.

We recall that a surface in 3D space is generally given either by an implicit equation of the form $F(x,y,z)=0$ or by a parametric presentation of the form $\left(x(u,v),y(u,v),z(u,v)\right)$ where *u* and *v* are real parameters.

A Computer Algebra System (CAS) is a multi-purpose package for doing mathematics, but not only for practical computations. Following Artigue in [2], we expect from software and computational tools to be pedagogical instruments for the learning of mathematical knowledge. The tools should help to avoid practices too much orientated towards pure lecturing or the procedural work, and rather push towards more profound understanding and acquisition of new mathematical skills. Using Artigue's taxonomy, working with technology should not be aimed at achieving pragmatic value (procedural technical work), but also and probably first epistemic value (mathematical understanding).

This allows to explore, to experiment, thus to enhance mathematical thinking by connecting mathematical fields which are sometimes viewed as totally separated. Cuoco and Levasseur emphasize in [6] that one way to do so is to address classical topics. Envelopes are one of them. Moreover, because of their applications, a study of envelopes may contribute to the integrated aspect of STEM education.

A CAS enables one to study envelopes in an either analytic or algebraic framework, and in both simultaneously. When the defining formula for the family is polynomial, then the subsequent equations are all polynomial. Algorithms employed to solve the non-linear systems of equations are based on Gröbner bases computations, described for example in [1] and [5]. Other algorithms exist to solve non-polynomial systems of equations. The power of such algorithms to solve geometric problems is emphasized in Pech's book [16].

Before proceeding further, we recall that when a parametric presentation exists for a given curve (or a surface), it is non-unique. For example, if the curve *C* is given by the presentation $\left(\boldsymbol{x},\boldsymbol{y}\right)=\left(\boldsymbol{f}_1(\boldsymbol{t}),\ \boldsymbol{f}_2(\boldsymbol{t})\right), \boldsymbol{t}\in \boldsymbol{I}$, where I is an interval in the set of real numbers, then setting $\boldsymbol{t}=\boldsymbol{g}(\boldsymbol{u}), \boldsymbol{u}\in \boldsymbol{J}$, where *J* is an interval and g a bijective function from J to I, then we obtain another parametrization for *C*.

The easiest example is given by the unit circle in the plane, which may be described by a trigonometric presentation such as $\left(x,y\right)=\left(\cos t,\sin t\right), t\in\left[0,2\pi\right]$ and also by a rational parametrization such as $\left(x,y\right)=\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right), t\in \mathrm{R}$. The usage of rational parametrization leads to polynomial equations, whence allows the usage of the algorithms based on Gröbner bases computations. The examples of this paper are given in such a setting.

The output of these computations is a parametric representation of an envelope. At this step, it is already possible to plot some elements of the family together with the envelope which has been found. A further computational step consists in eliminating the parameter in order to obtain an implicit equation for the envelope. This implicitization is not always possible. In [18], Schultz and Juttler explain how to have an approximate implicitization, necessary for numerous applications of envelopes in engineering and industry. When implicitization is possible, there may be surprising results, revealed by the usage of the graphical features of the CAS. We study such an example in Section III. We refer also to Peternell and Pottman [17].

A central feature used for the study of envelopes in 2D is the existence of a slider bar, which provided an interactive representation of the family of curves, not only an animated representation (we refer to the "animate" option generally available with the plot commands and the implicit plot commands of a CAS, not to the possibility of human interaction with the computer output using the mouse). Examples of the output of sessions based on the usage of the slider bar are displayed and commented in [7]). Non-availability of a slider in 3D is a pitfall, but as mentioned previously, other dynamical features may be available, generally less efficient for visualization of an envelope. We used the CAS ability to plot surfaces in 3D space and to rotate plots dynamically using the mouse.

In this paper, we study envelopes of 1-parameter families and 2-parameter families of planes in 3-dimensional space. Using Duval's taxonomy, working in an algebraic register should be similar to what happened with plane curves, but the work in the graphical register will encounter specific problems. Visualization in 3D space uses algorithms different from those in usage for 2D plots, and there exist various commands based on different algorithms for plotting surfaces. For example, Maple has a command **plot3d** which enables to plot graphs of functions given by an analytic formula of the form $z = f(x,y)$ and also graphs given by a parametric presentation. The command **implicitplot3d** enables to plot a surface given by an implicit equation of the form $f(x,y,z)=0$. Each of the commands includes a different choice for the mesh (triangular for **implicitplot3d**, based on geodesics, etc. for **plot3d** in a parametric setting), therefore for a surface given by different presentations, either implicit or parametric, the output may look quite different. Moreover, additional options have an influence on the output and the user has to develop specific skills to use these options. It follows that the problems a student has to deal with when representing on a screen a non-planar object may be non-trivial. Such issues have been addressed by in [22] and [23].

For the automated part of the work, we used the Maple package.

2. A short reminder on envelopes of 1-parameter families of surfaces

A set of surfaces $\{S_t\}$ in 3-dimensional space depending on a real parameter *t* is called a 1-parameter family of surfaces. In what follows, we suppose that all the surfaces S_t are smooth. Kock distinguishes in [13] and [14] three alternative ways to define an envelope *E* for such a family:

1. Synthetic: *E* is *the* union of *the* characteristics; the characteristic C_t is *the limit curve* of the family of curves $S_t \cap S_{t+h}$ as $h \to 0$.
2. Impredicative: *E* is a surface with the property that at each of its points, it is tangent to a unique surface from the given family (the locus of points where *E* touches S_t).
3. Analytic: assume that there exists a function $F(x,y,z,t)$ such that for each *t*, S_t is the zero set of $F(x,y,z,t)=0$. Then the surface *E* is the union of the *F*-discriminant curves, where the *F*-discriminant curve C_t^F for the parameter value *t* is the solution set for the *F*-discriminant system of equations

$$(*) \quad \begin{cases} F(x,y,z,t)=0 \\ \dfrac{\partial F}{\partial t}(x,y,z,t)=0 \end{cases}.$$

As Kock notes, the two first approaches are purely geometric and reveal some problems. For example, the notion of a *limit curve* in the synthetic definition is not well-defined (in which space do we work? with which topology?). Nevertheless, it is often used; see for example Eiden's book [10] p.232 (construction of Steiner's hypocycloid as an envelope of a specific family of lines). Some authors show how to derive from the synthetic definition the system of Equations (*).

In the impredicative definition, uniqueness is not clear, and no computational method is given. Finally, the analytic definition imposes the knowledge of the function *F* and the study should include checking that another function defining the same family yields the same envelope.

3. A 1-parameter family of planes

Consider the family of planes given by the equation $x+ty+t^2z=t^3$, where *t* is a real parameter. Our goal is to determine whether this family has an envelope and what is this envelope. It is difficult to see in Figure 1 the intersection points of the planes, and even to imagine the envelope, if there is one.

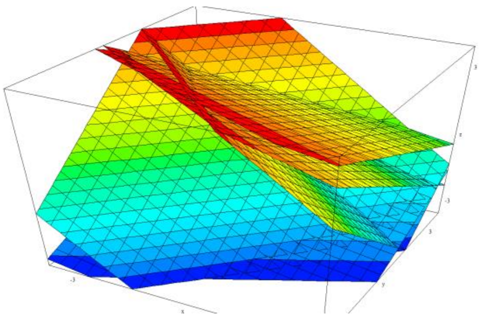


Figure 1: Visualizing the family of planes whose equation is $x+ty+t^2z=t^3$

For Figure 1, we chose $c=-2,-1,1,1.5,2$. Conjecturing here what the envelope of the family is impossible.

1. The envelope of the family: synthetic method.

Let us consider two planes in the family, namely with equations $x+ty+t^2z=t^3$ and $x+(t+\varepsilon)y+(t+\varepsilon)^2z=(t+\varepsilon)^3$. The intersection of these planes is given by

$$\begin{cases} x=-t(t+\varepsilon)(2t+\varepsilon-c) \\ y=\varepsilon^2+\varepsilon(3t-c)+t(3t-2c), \\ z=c \end{cases}$$

where *c* is a real parameter.

Computing the limit for ε arbitrary close to 0, we obtain

$$(1)\ \begin{cases} x=t^2(c-2t) \\ y=t(3t-2c),\ c,t\in\mathrm{R}. \\ z=c \end{cases}$$

This parametrization represents a surface in the three-dimensional space. The display in Figure 2 is the graphical output obtained when working with a CAS. The output itself suggests that this surface is a *ruled surface*.

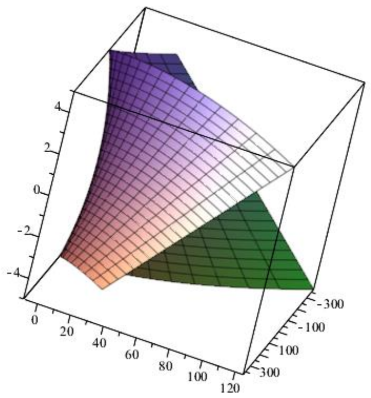


Figure 2: The envelope plotted by the CAS: a ruled surface?

Actually, this is a special case of a general situation:

Theorem: If a parameterized family of planes has an envelope, then this envelope is a ruled surface.

Proof:

Consider a 1-parameter family of plane given by the equation

$$a(t)x + b(t)y + c(t)z - 1 = 0\,,$$

where the coefficients are real functions of the real parameter t. The system of equations (*) is here:

$$\begin{cases} a(t)x + b(t)y + c(t)z - 1 = 0 \\ a'(t)x + b'(t)y + c'(t)z \quad = 0 \end{cases}.$$

This system describes the intersection of two planes. If this intersection is non-empty, it is a line. Therefore, if it exists, the envelope is generated by lines. It is a ruled surface. ■

It can be easily proven that if the given family of planes has a fixed line, then an envelope does not exist.

The envelope of the family: analytic method.

We use again the Computer Algebra System to solve the system of equations

$$(4)\begin{cases} f\left(x,y,z,t\right) = 0 \\ \dfrac{\partial}{\partial t} f\left(x,y,z,t\right) = 0 \end{cases}.$$

Here we have: $\begin{cases} x + ty + t^2 z - t^3 = 0 \\ y - 2tz - 3t^2 = 0 \end{cases}$.

The system can be solved either by hand or using a CAS. The solution is given by:

$$(5) \qquad \begin{cases} x = t^2(z - 2t) \\ y = t(3t - 2z)\,. \\ \quad z \in \mathrm{R} \end{cases}$$

Actually this is exactly System (3), which we could expect.

Now we address the implicitization issue. First, we perform the work by hand. To eliminate *t* between the first two equations, we solve the equation $y = t(3t-2z)$ for *t*, and obtain two solutions: $t = \frac{1}{3}\left(z - \sqrt{3y+z^2}\right)$ and $t = \frac{1}{3}\left(z + \sqrt{3y+z^2}\right)$.

Then we substitute the two solutions in the equation $x - t^2(z-2t) = 0$:

$$x - \left(\frac{1}{3}\left(z - \sqrt{3y+z^2}\right)\right)^2 \left(z - \frac{2}{3}\left(z - \sqrt{3y+z^2}\right)\right) = 0$$

$$x - \left(\frac{1}{3}\left(z + \sqrt{3y+z^2}\right)\right)^2 \left(z - \frac{2}{3}\left(z + \sqrt{3y+z^2}\right)\right) = 0$$

We denote the solutions of these equations by x_1 and x_2. Expanding the equation $27(x-x_1)(x-x_2) = 0$, we obtain an implicit equation for the envelope:

$$(6) \quad 27x^2 + 18xyz + 4xz^3 - 4y^3 - y^2z^2 = 0.$$

Automated derivation of the result:

We display now the code for a Maple session (only the most important part; file [S1] contains more than that, in order to show details for students):

restart: with(plots): with(PolynomialIdeals):

$F := x + t\cdot y + t^2\cdot z - t^3;\ derF := \frac{d}{dt}F;$

par := solve({F = 0, derF = 0}, {x, y, z}):

f1 := subs(z = u, rhs(par[1])): f2 := subs(z = u, rhs(par[2])): f3 := subs(z = u, rhs(par[3])):

plot3d([f1, f2, f3], t = -2 .. 2, u = -2 .. 2, axes = boxed);

The last command yields the plot in Figure 5b. In the first row, variables are reset and the needed packages are uploaded. Then a parametric implicit equation for the family is entered. Next commands are aimed at writing and solving System (4). The solution is obtained in parametric form. In order not to use the variable *z* as a parameter for the surface (the envelope), a substitution is necessary (a change of variable). Finally, the plot is displayed.

In what follows, an implicit equation for the envelope *E* is derived (also in [S1]).

$J := \langle x - f1, y - f2, z - f3 \rangle$:

JE := EliminationIdeal(J, {x, y, z});

$$\langle -27x^2 - 18yxz + 4y^3 - 4xz^3 + z^2y^2 \rangle$$

GJE := Generators(JE); g := GJE[1]:

implicitplot3d(g=0,x=-3..3,y=-3..3,z=-3..3,axes=boxed,numpoints=2000);

The last command produces the plot in Figure 3.

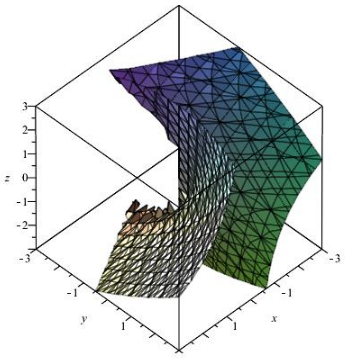


Figure 3: The envelope obtained with implicit plot

Fortunately, implicitization yields here the same result as obtained by hand. Note that the plot obtained with the **implicitplot3d** command is of much lower quality than the parametric plot obtained previously. This issue has been discussed in [22] and [23]: **implictplot3d** uses a *standard* triangular mesh to plot the surface, whence the "teeth" close to the singular points, but the parametric plot chooses a mesh using specific features of the surface (e.g. isoclines). In Figure 5b, the parametric plot uses the fact that the envelope is a ruled surface.

The envelope E of the family is determined by implicit equation (6). Denote by F the left-hand side of this equation, namely $F(x,y,z)=27x^2+18xyz+4xz^3-4y^3-y^2z^2$. The set of singular points of E is the set of points where $\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z}=0$. We have:

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}=\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{y}}=\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{z}}=0 \Leftrightarrow \begin{cases} 54\boldsymbol{x}+18\boldsymbol{yz}+4\boldsymbol{z}^3=0 \\ 18\boldsymbol{xz}-12\boldsymbol{y}^2-2\boldsymbol{yz}^2=0 \\ 18\boldsymbol{xy}+12\boldsymbol{xz}^2-2\boldsymbol{y}^2\boldsymbol{z}=0 \end{cases} \Leftrightarrow \begin{cases} 27\boldsymbol{x}+9\boldsymbol{yz}+2\boldsymbol{z}^3=0 \\ 9\boldsymbol{xz}-6\boldsymbol{y}^2-\boldsymbol{yz}^2=0 \\ 9\boldsymbol{xy}+6\boldsymbol{xz}^2-\boldsymbol{y}^2\boldsymbol{z}=0 \end{cases},$$

and the solutions are given by: $(x,y,z)=\left(\frac{1}{27}t^3,-\frac{1}{3}t^2,t\right), t\in \mathbb{R}$.

This is a parametric presentation of a space curve, all of its points being singular points of E. It is called a cuspidal curve of E; see two views in Figure 4, obtained with the **spacecurve** command.

It is possible to find this cuspidal edge using automated methods, as follows:

with(*VectorCalculus*) :
grg := Gradient(g, [x, y, z]);

$$(-54x-18zy-4z^3)\bar{e}_x+(-18xz+12y^2+2z^2y)\bar{e}_y+(-18yx-12xz^2+2zy^2)\bar{e}_z$$

solve({*grg*[1]=0,*grg*[2]=0,*grg*[3]=0},{*x*,*y*,*z*});

$$\left\{x=\frac{1}{27}z^3, y=-\frac{1}{3}z^2, z=z\right\}$$

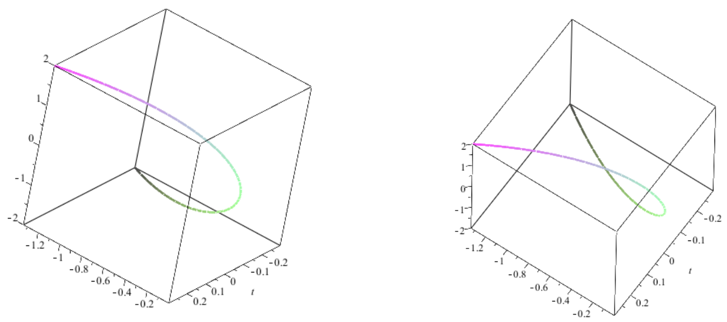
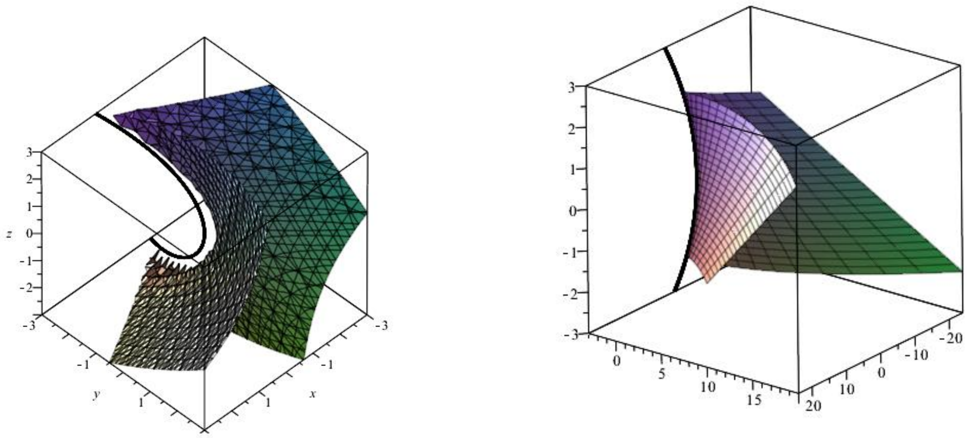


Figure 4: the cuspidal edge of the envelope

If we wish to plot the envelope using **implicitplot3d** together with emphasizing the cuspidal edge, we obtain Figure 5a, giving the impression that the curve is not the good one. The problem comes, once again, from the fact that the curve is a locus of singularities and the above command has hard time in the neighbourhood of singularities.



(a) **A problematic plot** **(b) an accurate plot.**

Figure 5: the envelope and the locus of singular points

If we used a parametric plot, then the display is much more accurate, as in Figure 5b.

4. A 2-parameter family of planes

We consider now a family $\{S_{u,v}\}$ of surfaces defined by the equation $F(x,y,z,u,v)=0$, where u and v are real parameters. Here the usage of the synthetic definition is still less rigorous than in the 1-parameter case. We work analytically. If it exists, an envelope of the family $\{S_{u,v}\}$ is determined by the system of equations:

$$(**)\qquad \begin{cases} F(x,y,z,u,v)=0 \\ \dfrac{\partial F}{\partial u}(x,y,z,u,v)=0 \\ \dfrac{\partial F}{\partial v}(x,y,z,u,v)=0 \end{cases}.$$

The proof follows the same pathway as before.
Take now the 2-parameter family of planes given by the equation

$$(7) \quad x+(u+v)y+\left(u^2+v^2\right)z-\left(u^3+v^3\right)=0,$$

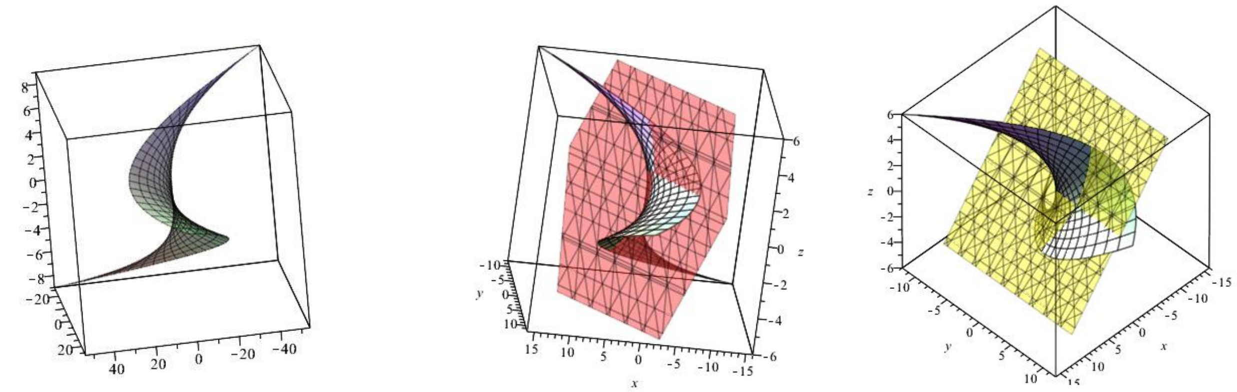
where u and v are real parameters. The envelope is determined by the following system of equations:

$$(8) \quad \begin{cases} x+(u+v)y+\left(u^2+v^2\right)z-\left(u^3+v^3\right)=0 \\ y+2uz-3u^2=0 \\ y+2vz-3v^2=0 \end{cases}$$

whose solutions are given by:

$$(9) \quad \begin{cases} x=\dfrac{1}{2}(u+v)\left(u^2-4uv+v^2\right) \\ y=-3uv \\ z=\dfrac{3}{2}(u+v) \end{cases}.$$

Equations (9) are a parametric presentation of a surface, displayed in Figure 6a. Figures 6b and 6c show the tangency of the surface with one of the plane in the family.



(a) The envelope **(b) With the plane $(u,v)=(1,0)$** **(c) With the plane $(u,v)=(1,-1)$**

Figure 6: Envelope of a 2-parameter family of planes

As for the 1-parameter case, we may check existence of singular points. Now denote:

$$\vec{r}(u,v)=\left(\frac{1}{2}(u+v)\left(u^2-4uv+v^2\right),\ -3uv,\ \frac{3}{2}(u+v)\right).$$

A point on E is singular if the vectors $\dfrac{\partial \vec{r}}{\partial u}$ and $\dfrac{\partial \vec{r}}{\partial v}$ are linearly dependent. We have:

$$\frac{\partial \vec{r}}{\partial u}=\left(-\frac{3}{2}u^2+3uv+\frac{3}{2}v^2,-3v,\frac{3}{2}\right) \quad \text{and} \quad \frac{\partial \vec{r}}{\partial v}=\left(-\frac{3}{2}u^2+3uv+\frac{3}{2}v^2,-3u,\frac{3}{2}\right)$$

These vectors are linearly dependent if, and only if, $u=v$. By substitution of $v=u$ into System (9), we obtain a parametric representation of a space curve C:

$$(10) \quad \begin{cases} \boldsymbol{x}=-2\boldsymbol{u}^3 \\ \boldsymbol{y}=-3\boldsymbol{u}^2,\ \boldsymbol{u}\in \mathrm{R}. \\ \boldsymbol{z}=3\boldsymbol{u} \end{cases}$$

The curve is shown in Figure 7; the leftmost plot is the curve only, and the two others show the curve and the surface. Note that the envelope appears as *a variety with boundary C* in $\mathbf{R}^3$. The boundary is exactly the curve of singular points we found previously.

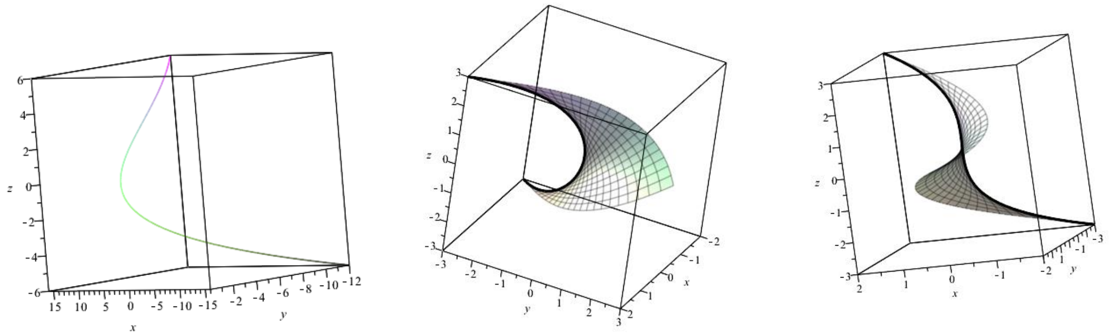


Figure 7: The locus of singular points of the envelope

In this example, checking where the vectors $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ are linearly dependent was easy. For the sake of more advanced examples, we give here the Maple code we used (file [S2]):

$restart : with(plots) : with(LinearAlgebra) :$ $F := x + (u+v)\cdot y + \left(u^2+v^2\right)\cdot z - \left(u^3+v^3\right) :$ $duF := \frac{d}{du} F : dvF := \frac{d}{dv} F :$ $parenv := solve(\{F=0, duF=0, dvF=0\}, \{x,y,z\}) :$	
$low := 1 : high := 1 : plot3d([rhs(parenv[1]), rhs(parenv[2]),$ $rhs(parenv[3])], u=-low..high, v=-low..high, axes=boxed,$ $labels=[x,y,z]) : p1 := \% : plot3d([rhs(parenv[1]),$ $rhs(parenv[2]), rhs(parenv[3])], u=-low..high, v=-low..high,$ $axes=boxed, labels=[x,y,z], transparency=0.6) : p2 := \% :$	
$subs(u=1, subs(v=-1, F)) : implicitplot3d(\%=0, x=-10..10, y=-10$ $..10, z=-6..6, color=yellow, transparency=0.5) : pl1 := \% :$ $subs(u=1, subs(v=0, F)) : implicitplot3d(\%=0, x=-10..10, y=$ $-10..10, z=-6..6, color=red, transparency=0.5) : pl2 := \% :$	Two plots: one is opaque for a general display, and one is transparent for the display of the locus of singular points, drawn on the surface.
$display(p1, pl1) :$ $Vu := \left\langle \frac{d}{du}(rhs(parenv[1])); \frac{d}{du}(rhs(parenv[2])); \frac{d}{du}(rhs(parenv[3]))\right\rangle :$ $Vv := \left\langle \frac{d}{dv}(rhs(parenv[1])); \frac{d}{dv}(rhs(parenv[2])); \frac{d}{dv}(rhs(parenv[3]))\right\rangle :$ Vu &x $Vv :$ $spacecurve\left([-2\cdot u^3, -3\cdot u^2, 3\cdot u], u=-2..2, axes=boxed, labels=[x,y,z]\right) :$	Computes the 1st derivatives of the vector, then their cross product in order to check their linear dependence.

$spacecurve\left(\left[-2 \cdot u^3, -3 \cdot u^2, 3 \cdot u\right], u = -low..high, axes = boxed, labels = [x, y, z], color = black, thickness = 2\right) : sing := \% :$ $display(p2, sing) :$	

Now we address the implicitization issue. Here is the Maple code (at the end of file [S2]):

$with(PolynomialIdeals) :$

$J := \langle x - rhs(parenv[1]), y - rhs(parenv[2]), z - rhs(parenv[3]) \rangle :$

$JE := EliminationIdeal(J, \{x, y, z\}) :$ # gives the implicit equation for the surface

$eqenv := Generators(JE)[1] :$ #transforms the output in a from which may be used by the next command

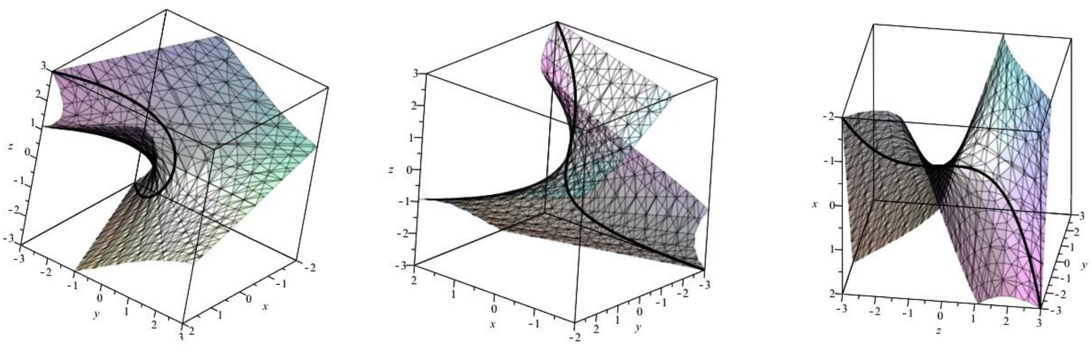
$implicitplot3d(eqenv = 0, x = -2..2, y = -3..3, z = -3..3, axes = boxed, numpoints = 2000, transparency = 0.5) : p3 := \% :$

$display(p3, sing);$ #displays the obtained surface together with the curve of singular points

Denote by S the surface whose implicit equation has been obtained in this session, namely:

$$S : 27x + 18zy + 4z^3 = 0 . \tag{11}$$

Three views of the output of the last command are shown in Figure 8:



(a) (b) (c)

Figure 8: the implicit equation surface with the curve *C*

Let us analyze the situation:

- Figure 8a is sufficient to show that *C* is drawn on *S*; this is proven by a substitution of Equations (1) into Equation (11).
- Figure 8c shows that S apparently has a saddle structure. This is proven by standard Calculus methods, but these methods will show that *S* has only one singular point, the saddle point.
- The 1-parameter family of the previous section is a subfamily of the 2-parameter family we study here, obtained for $v = 0$. Intuitively, we would have expected to see here too the cuspidal edge of the previous section, but we have it not.

Actually, superposition of the rightmost plot in Figure 7 with Figure 8b gives the clue, as shown in Figure 9:

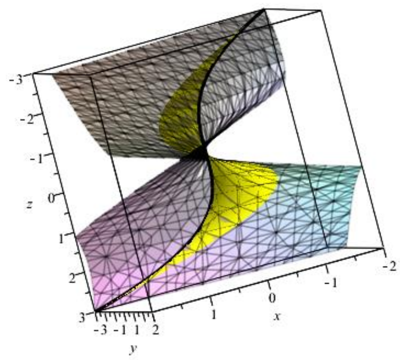


Figure 9

In the implicitization process Equations (9) imply Equations (11), but the converse is not true. The surface *S* is only "half" the surface in Figure 8, and the curve *C* is its boundary.

5. Conclusions

Dana-Picard and Zehavi discussed in [8] the transition from 1-parameter families of plane curves to 1-parameter families of surfaces. The assumption was that the study of a geometric progression family of curves in the plane, then to a geometric progression family of planes in the 3-dimensional space may lead a student to understand that he/she is discovering the bases of a general theory. Here we study a transition in a 3D setting, from a 1-parameter situation towards a 2-parameter situation. We consider the two transitions as successive loops in Buchberger's educative spiral (see [3]), applied to the theory of envelopes. The level of abstraction increases with the transition from surfaces to 1-parameter families of surfaces, and again when going further to 2-parameter families of surfaces.

Among Kock's three definitions of an envelope of a 1-parameter family of surfaces ([13], [14], and v.s. Section 2), we used at first the synthetic one, which enables the derivation of automated proofs, as in [8] for families of plane curves. Despite the fact that the "space of plane curves" is not well-defined, neither is a topology in this space, the intuition provided by the CAS, when showing two "infinitesimally close" objects in the family and providing a visualization of the limit process thus building a characteristic curve, was very efficient. After all, this is often the way the tangent to a curve at a given point is introduced to high-school students: plot the curve C and a fixed point A on *C*, and a mobile point *M* on *C*. Draw the line (*AM*) and push the point *M* along *C* towards *A* until *A* and *M* coalesce.

This is true for the plane, less for surfaces in space as visualization is more problematic. Figure 4 shows a couple of planes in the geometric progression family of planes; this plot cannot support intuition. This is totally different from what happens in the corresponding plane case.

Besides being examples showing common features and different features, the two examples which have been presented here have another mathematical connection. The envelope of a 1-parameter family of spheres is not a ruled surface, but a relation exists, via a Lie transformation, between this envelope and the ruled surface generated by the lines corresponding to the spheres in this transformation (see [11]). Such a remark justifies further work, another loop on Buchberger's spiral.

Oldknow and Tetlow explain in [15] that "Teaching 3D geometry beyond primary school level has presented real problems to many teachers, especially when pupils, and teachers, have very limited spatial awareness themselves". The authors teach pre-service and in-service teachers, and also students towards a degree in engineering. For them, Oldknow and Tetlow's remark is still valid:

adults may have limited skills to visualize a 3D situation. The development of dynamical software for 3D geometry is really important. It will support experimentations and enable to conjecture what will be proven afterwards using joint methods, paper-and-pencil based and automated.

Back to Thom's remark in [19], the usage of technology enables to re-introduce envelopes into the curriculum. Working in a mixed framework, using paper-and-pencil methods together with CAS and DGS helps students to develop technological skills. The study of concrete examples from the fields mentioned in the introduction makes the topic important in STEM education.
We wish to finish with a quote from Hilbert's address in 1900 on the importance of envelopes (in [12]): "who would give up the picture of a family of curves or surfaces with its envelope which plays so important a part in differential geometry, in the theory of differential equations, in the foundation of the calculus of variations and in other purely mathematical sciences?"

8. References

[1] Adams, W. and Loustaunau, P., *An Introduction to Gröbner Bases*, Graduate Studies in Mathematics 3, American Mathematical Society, RI: Providence, 1994.

[2] Artigue, M., *Learning Mathematics in a CAS Environment: The Genesis of a Reflection about Instrumentation and the Dialectics between Technical and Conceptual Work*, International Journal of Computers for Mathematical Learning 7, 245-274, 2002.

[3] Buchberger B., *Teaching without Teachers?*, Talk at VISIT-ME 2002, Vienna, July 10, 2002, available: http://rfdz.ph-noe.ac.at/fileadmin/Mathematik_Uploads/ACDCA/VISITME2002/contribs/Buchberger/Buchberger.PDF

[4] Cagnac G., Ramis E. and Commeau J., *Nouveau cours de Mathématiques Spéciales*, Masson, Paris, 1966.

[5] Cox, D., Little, J. and O'Shea, D., *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Undergraduate Texts in Mathematics, Springer Verlag, New York, 1992.

[6] Cuoco, A. and Levasseur, K., *Classical Mathematics in the Age of CAS*. In *Computer Algebra System in Secondary School Mathematics Education*, edited by Jim Fey et al, pp. 97-116. Reston Va.: NCTM, 2003.

[7] Dana-Picard Th. and Zehavi, N., *Revival of a classical topic in Differential geometry: the exploration of envelopes in a computerized environment*, International Journal of Mathematical Education in Science and Technology 47(6), 938-959, 2016.

[8] Dana-Picard Th. and Zehavi N., *Envelopes in a computerized environment: the transition from 2D to 3D*, In E. Nardi, C. Winsløw & T. Hausberger (Eds.), Proceedings of the First Conference of the International Network for Didactic Research in University Mathematics (INDRUM 2016, 31 March-2 April 2016) (pp. 370-379). Montpellier, France: University of Montpellier and INDRUM, 2016.

[9] Duval, R., *A cognitive analysis of problems of comprehension in a learning of mathematics*, Educational Studies in Mathematics 61, 103-131, 2006.

[10] Eiden, J.-D., *Géométrie analytique classique*, Calvage et Monnet, Paris, 2009.

[11] Hávlíček, K., Sur les Surfaces Enveloppes de Sphères (On enveloping surfaces of spheres), Časopis pro pěstování matematiky a fysiky, roč. 74, 21-40, 1949. Available: http://dml.cz/dmlcz/109145

[12] Hilbert, D., Talk at International Congress of Mathematicians, Paris, 1900. Available: http://www-history.mcs.st-and.ac.uk/Extras/Hilbert_Problems.html

[13] Kock A. (2003). *Envelopes – notion and definiteness*, Aarhus Mathematics Institute Preprint Series (12). Available: http://home.math.au.dk/kock/envelope2.pd

[14] Kock, A., *Synthetic Differential Geometry*, Cambridge University Press, Cambridge, U.K., 2006.

[15] Oldknow, A. and Tetlow, L., *Using dynamic geometry software to encourage 3D visualisation and modelling*, The Electronic Journal of Mathematics and Technology 2 (1), 54-62, 2008.
[16] Pech, P., *Selected Topics in Geometry with Classical vs. Computer Proving*, World Scientific, 2007.
[17] Pottman, H. and Peternell M., Envelopes – Computational Theory and Applications, in *Proceedings of Spring Conference in Computer Graphic*, Budmerice, Slovakia, 3-23, 2010.
[18] Schultz, T. and Juttler, B., Envelope Computation by Approximate Implicitization, *FSP Report 106, Forschungswchwerpunkt S92,* 2010. Retrieved on May 13th, 2014: http://www.industrial-geometry.at/uploads/nm_report106.pdf
[19] Thom, R., Sur la théorie des enveloppes, *Journal de Mathématiques Pures et Appliquées* 41 (2), 177-192, 1962.
[20] Trigueros-Gaismann, M. and Martinez-Planell, R., Geometrical Representations in the Learning of two Variable Functions, *Educational Studies in Mathematics* 81(3), 365-384, 2012.
[21] Yerushalmy, M., Designing representations: Reasoning about functions of two variables, *Journal for Research in Mathematics Education*, 28, 431-466, 1997.
[22] Zeitoun, D. and Dana-Picard, Th., *Accurate visualization of graphs of functions of two real variables*, International Journal of Computational and Mathematical Sciences 4(1), 1-11, 2010.
[23] Zeitoun, D. and Dana-Picard, Th., *Zooming algorithms for accurate plotting of functions of two real variables*, to appear in 'Applications of Computer Algebra 2015: Kalamata, Greece', I. Kotsireas and E. Martinez-Moro (edts), Springer Proceedings in Mathematics & Statistics, 2016.

9. Supplementary Electronic Materials

[S1] Maple file: **1-param family of planes.mw**
[S2] Maple file: **2-param family of planes.mw**